Operator content of n-state quantum chains in the c=1 region

1988 J. Phys. A: Math. Gen. 21 2805

(http://iopscience.iop.org/0305-4470/21/12/022)

J. Phys. A: Math. Gen. **21** (1988) 2805-2824. Printed in the UK

Operator content of *n*-state quantum chains in the $c=1$ region

G von Gehlen†, V Rittenberg‡ and G Schütz†

† Physikalisches Institut, Universität Bonn, D-5300 Bonn 1, Federal Republic of Germany
‡ Centro Brasileiro de Pesquisas Fisicas—CBPF/CNPq, Rua Dr Xavier Siguad 150, 22290 Rio de Janeiro, RJ, Brazil

Received 12 October 1987, in final form 16 February 1988

Abstract. We conjecture the operator content for n-state quantum chains ($n \geqslant 5$) in the domains of the coupling constants where the central charge of the Virasoro algebra is equal to one. Free boundary conditions as well as boundary conditions compatible with the torus are considered. The conjectured operator content is compared with finite-size scaling estimates.

1. Introduction

This paper is a continuation of our previous studies on the critical behaviour of n-state quantum chains. These chains are defined by the Hamiltonians

$$H = -\frac{1}{\xi}\sum_{j=1}^{N}\sum_{k=1}^{n-1} a_k(\sigma_j^k + \lambda\Gamma_j^k\Gamma_{j+1}^{n-k}) \tag{1.1}$$

where $a_k = a_{n-k}$ are real coupling constants, λ plays the role of the inverse of the temperature, N represents the number of sites and the $n\times n$ matrices σ and Γ are

$$\sigma = \begin{bmatrix} \omega^0 & 0 & \cdots & 0 \\ 0 & \omega^1 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \omega^{n-1} \end{bmatrix}$$
$$\Gamma = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}. \tag{1.2}$$

Here $\omega = \exp(2\pi i/n)$ and ξ is a normalisation factor which fixes the timescale, to be discussed later. The Hamiltonian H is self-dual, i.e.

$$H(\lambda) = \lambda H(1/\lambda). \tag{1.3}$$

The cases $n=2$ and $n=3$ correspond to the Ising and three-state Potts model and their operator content is known. The case $n=4$ which describes the Ashkin-Teller model has also been recently understood (Rittenberg 1987, Baake *et al* 1987b, c, Yang 1987a, b, Yang and Zheng 1987, Saleur 1987). Starting with $n=5$ the critical properties of the system are more complex and several values of the central charge of the Virasoro algebra occur for the same value of n. Roughly speaking one expects the central charge which measures the number of degrees of freedom of the system to increase with n and this indeed happens for certain values of the coupling constants. At the same time there are regions in the space of the coupling constants where some degrees of freedom are frozen and the central charge is smaller. From older work (José *et al* 1977, Elizur *et al* 1979, Cardy 1978, Kadanoff 1979, Fradkin and Kadanoff 1980, Nienhuis 1984) and our own numerical studies, all the systems with $n \geqslant 4$ have a domain of the coupling constants where $c=1$ and other domains where c is larger. Some partial results for the six- and eight-state models have been already published (von Gehlen and Rittenberg 1986a, 1987, Schütz 1987). For example, for the choice of parameters $a_k = 1$ for k odd, $a_k = 0$ for k even, we obtained $c \approx 1.25$ for $n=6$ and $c \approx 1.30$ for $n=8$. For $a_k = 1/\sin(\pi k/n)$ one has $c = 2(n-1)/(n+2)$ (Zamolodchikov and Fateev 1985, Alcaraz 1986) whereas for the vector-Potts case $a_2 = \ldots = a_{n-2} = 0$, $a_1 = a_{n-1} = 1$ our numerical analysis gives $c=1$ for $n \geqslant 5$. For other values of the a_k still other values of c appear.

In this present paper we confine ourselves to the domain in the space of the coupling constants where $c=1$. It turns out that, although the quantum chains defined by equation (1.1) have only the discrete dihedral group D_n as global symmetry, at criticality and large N the symmetry is $U(1) \times U(1)$ for boundary conditions compatible with the torus and $U(1)$ for free boundary conditions. As a result the operator content of these models can be expressed in terms of irreducible representations of two commuting $U(1)$ Kac-Moody algebras for the torus and one $U(1)$ Kac-Moody algebra for free boundary conditions (see Baake *et al* (1987b, c) and references therein for the $n=4$ case). The higher symmetry at criticality and the value of the central charge $c=1$ suggest that the dimensions of the primary fields of the model are given by the Gauss model (di Francesco *et al* (1987) and references therein):

$$\Delta = (M \bullet gN)^2/(4ng) \tag{1.4}$$

where M and N are integers and g is a parameter. There are some sectors of the models which are not described by equation (1.4). In these sectors one gets

$$\Delta = \tfrac{1}{16} + m \tag{1.5}$$

where m is integer or half-integer. This corresponds to an irreducible representation of a twisted $U(1)$ Kac-Moody algebra to be defined later.

We now have to explain how the parameter g is related to the physics of the problem. Let us assume that for a certain choice of the coupling constants a_k and $\lambda = 1$ (the self-dual line) one finds $c=1$. Then the system stays critical in a domain of λ, called the critical fan:

$$1/\lambda_{\max} \leqslant \lambda \leqslant \lambda_{\max}. \tag{1.6}$$

Now $g = g(\lambda)$ turns out to be a monotonic function of λ such that

$$g(1/\lambda_{\max}) = 4/n \qquad g(1) = 1 \qquad g(\lambda_{\max}) = \tfrac{1}{4}n. \tag{1.7}$$

The form of the function $g(\lambda)$ depends on the coupling constants a_k. This coincides with the known result of the clock model (see, for example, Fradkin and Kadanoff (1980); their coupling constant K_{eff} is related to our g by the relation $ng = 2\pi K_{\text{eff}}$).

To illustrate the picture let us consider the leading magnetic exponents x_Q (periodic boundary conditions) corresponding to the charge-Q sector of the theory ($Q = 1, 2, \ldots, n-1$). We find

$$x_Q = x_{n-Q} = 2\Delta_Q = Q^2/2ng \qquad Q = 1, 2, \ldots, [n/2] \tag{1.8}$$

which gives

$$2Q^2/n^2 \leqslant x_Q \leqslant \tfrac{1}{8}Q^2 \tag{1.9}$$

and we recover a known result (Elizur *et al* 1979, Cardy 1978).

Notice that, for $n = 4$, we see from equation (1.7) that $g \equiv 1$. Then we may have $\lambda_{\max} = \lambda_{\min} = 1$, which is only one point in the phase diagram of the Ashkin–Teller model. This point corresponds to a Kosterlitz–Thouless phase transition.

The aim of this paper is to present a complete description of the critical exponents of the n-state models. They are obtained from the finite-size limit spectra of the Hamiltonian with different boundary conditions. The free-boundary-condition case gives the surface exponents (Cardy 1984, 1986, von Gehlen and Rittenberg 1986a, b). There are $2n$ toroidal boundary conditions corresponding to the $2n$ elements of the group D_n. The boundary condition corresponding to the unit element (periodic boundary condition) gives the anomalous dimensions of the scalar operators (like the energy density) and the order operators, the other $2n-1$ boundary conditions corresponding to various disorder and parafermionic operators (Cardy 1986). We would like to stress that there are hypersurfaces in the space of the coupling constants a_k (see equation (1.1)) where the symmetry is larger than D_n (Marcu *et al* 1981). In these cases one has to consider more boundary conditions and determine the corresponding finite-size spectra. This very interesting exercise is not considered here.

Our paper is organised as follows. In § 2 we discuss the symmetry of the problem corresponding to different boundary conditions. In § 3 we define the finite-size scaling quantities which give the operator content of the model and we summarise the necessary knowledge on the representation theory of Virasoro and U(1) Kac–Moody algebras. In § 4 we consider the case of free boundary conditions. We first review the situation in the Ashkin–Teller model ($n = 4$) and then conjecture the operator content for $n \geqslant 5$. This conjecture is compared with numerical estimates on the self-dual line ($\lambda = 1$) for $n = 5$, 6, 8 and 12. The case of boundary conditions compatible with the torus is considered in § 5. We first conjecture the operator content for the whole massless phase. Then we specialise to the case $\lambda = 1$ where the operator content is independent of the coupling constants and takes a simpler analytical form. This operator content is then compared with numerical estimates. The conclusions of our work are given in § 6 where we also present the large n-limit of the model. In the appendix, for completeness, we review the known construction of the irreducible representations of the U(1) untwisted and twisted Kac–Moody algebras.

2. Symmetry of the Hamiltonian for various boundary conditions

In this section we study the symmetry of the Hamiltonian equation (1.1) for various boundary conditions (BC) and the resulting decomposition of the spectra into sectors.

We first consider free BC

$$\Gamma_{n+1} = 0 \tag{2.1}$$

and periodic BC

$$\Gamma_{N+1}=\Gamma_1 \tag{2.2}$$

and denote the corresponding Hamiltonians by H^{F} and H^0, respectively. Both H^{F} and H^0 are invariant under the global transformations

$$(\Gamma'_j)^m = A^{mn}\Gamma_j^n \tag{2.3}$$

where A is one of the $(n-1)\times(n-1)$ matrices Σ^l or $\Sigma^k C$ ($l, k=0, 1, \ldots, n-1$) and

$$\Sigma=\begin{pmatrix} \omega & 0 & \cdots & 0 \\ 0 & \omega^2 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix} \qquad C=\begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & & & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.4}$$

The matrices Σ^l and $\Sigma^k C$ (l, $k=0$, $1, \ldots, n-1$) form the dihedral group D_n with $2n$ objects. Let us write $n=2p+1$ for n odd and $n=2p+2$ for n even.

The group D_n has p two-dimensional representations

$$D_Q(\Sigma^l)=\begin{pmatrix} \omega^{Ql} & 0 \\ 0 & \omega^{-Ql} \end{pmatrix} \qquad D_Q(C)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Q=1,\ldots,p \tag{2.5}$$

and for n odd there are two one-dimensional representations

$$\begin{aligned} &D_{0,+}(\Sigma^l)=1 \qquad && D_{0,+}(C)=1 \\ &D_{0,-}(\Sigma^l)=1 \qquad && D_{0,-}(C)=-1. \end{aligned} \tag{2.6}$$

For n even we have two more one-dimensional representations:

$$\begin{aligned} &D_{n/2,+}(\Sigma^l)=(-1)^l \qquad && D_{n/2,+}(C)=1 \\ &D_{n/2,-}(\Sigma^l)=(-1)^l \qquad && D_{n/2,-}(C)=-1. \end{aligned} \tag{2.7}$$

Since H^{F} and H^0 are invaraint under the transformations (2.3), their spectra decompose into sectors $H^{\mathrm{F}}_{\Lambda^{(\alpha)}}$, $H^0_{\Lambda^{(\alpha)}}$, respectively, according to the irreducible representations $\Lambda^{(\alpha)}$ of D_n, equations (2.5)-(2.7) (for special choices of the coupling constants a_k in equation (1.1), the symmetry can be larger than D_n, but for the present discussion we consider general a_k). In order to simplify the notation, we shall make use of the fact that D_n is a semidirect product of Z_n and Z_2. For periodic BC we write H^0_Q for $Q\neq 0$ and $Q\neq\frac{1}{2}n$ where $H^0_Q=H^0_{n-Q}$ build together the two-dimensional representation D_Q, equation (2.5). If $Q=0$ we have the two one-dimensional representations $D_{0,+}$ and $D_{0,-}$ and we write $H^0_{0,+}$ and $H^0_{0,-}$, respectively. Similarly, for $Q=\frac{1}{2}n$ (n even) and the representations (2.7) we write $H^0_{n/2,+}$ and $H^0_{n/2,-}$. The signs $\pm$ correspond to the $D_{0,\pm}(C)=\pm 1$, $D_{n/2,\pm}(C)=\pm 1$. The case of free BC is completely analogous.

Now we proceed to the other boundary conditions compatible with the torus. If

$$\Gamma^k_{N+1}=B^{km}\Gamma^m_1 \tag{2.8}$$

we denote the corresponding Hamiltonian by H^B, where B is one of the matrices $\Sigma^{\tilde{Q}}$ or $\Sigma^R C$ ($\tilde{Q}$, $R=0, 1, \ldots, n-1$). In general the symmetry of H^B will be smaller than D_n. It is given by the group G_B ($\mathrm{G}_B\subseteq\mathrm{D}_n$) of those matrices A in equation (2.8) which commute with B. Now it is trivial to show that two Hamiltonians H^{B_1} and H^{B_2} corresponding to two BC B_1 and B_2 have the same spectrum if the group elements B_1

Table 1. Symmetry of the n-state Hamiltonian for various boundary conditions.

	Boundary condition	Group	Number of elements	Elements
	Σ^0	D_n	$2n$	$\Sigma^l, \Sigma^m C$ ($l, m = 0, 1, \ldots, n-1$)
	$\Sigma^{\tilde{Q}}$	Z_n	n	Σ^l ($l = 0, 1, \ldots, n-1$)
$n = 2p+2$	Σ^{p+1}	D_n	$2n$	$\Sigma^l, \Sigma^m C$ ($l, m = 0, 1, \ldots, n-1$)
	$\Sigma^R C$ ($R = 0, \ldots, n-1$)	$\mathrm{Z}_2 \times \mathrm{Z}_2$	4	$\Sigma^0, \Sigma^{p+1}, \Sigma^R C, \Sigma^{R+p+1} C$
$n = 2p+1$	$\Sigma^R C$ ($k = 0, \ldots, n-1$)	Z_2	2	$\Sigma^0, \Sigma^R C$

and B_2 belong to the same conjugacy class. For $n = 2p+1$ (odd) the $2n$ group elements of D_n form $p+2$ conjugacy classes:

$$\begin{aligned} &\{\Sigma^0\} \qquad \{\Sigma^k, \Sigma^{n-k}\} \qquad k = 1, 2, \ldots, p \\ &\{C, \Sigma C, \ldots, \Sigma^{n-1} C\}. \end{aligned} \tag{2.9}$$

If n is even, $n = 2p+2$, and there are two more conjugacy classes: the last class of equation (2.9) splits into

$$\{\Sigma C, \Sigma^3 C, \ldots, \Sigma^{n-1} C\} \qquad \{C, \Sigma^2 C, \ldots, \Sigma^{n-2} C\} \tag{2.10}$$

and in addition we also have $\{\Sigma^{n/2}\}$.

In table 1 we show the symmetry groups G_B corresponding to the various boundary conditions B. The spectrum of H^B can now be decomposed into sectors $H^B_{\Lambda_B^{(\alpha)}}$ according to the irreducible representations $\Lambda_B^{(\alpha)}$ of G_B. If $\tilde{Q} = \frac{1}{2}n$ (n even), we have the sectors $H_Q^{n/2}$ ($Q \neq 0, \frac{1}{2}n$), $H_{0,\pm}^{n/2}$ (corresponding to the representations $D_{0,\pm}$) and $H_{n/2,\pm}^{n/2}$ (corresponding to $D_{n/2,\pm}$ (equation (2.7)). If the boundary condition is $\Sigma^R C$, the symmetry is $\mathrm{Z}_2 \times \mathrm{Z}_2$ (for n even) and the sectors will be denoted by $H^{\Sigma^R C}_{\Sigma^{n/2} = \pm, \Sigma^R C = \pm}$. For n odd the symmetry is only Z_2 and the sectors are $H^{\Sigma^R C}_{\Sigma^R C = \pm}$.

To sum up, for n even we have the sectors

$$H_Q^{\tilde{Q}}, H^0_{0,\pm}, H^0_{n/2,\pm}, H^{n/2}_{0,\pm}, H^{n/2}_{n/2,\pm}, H^{\Sigma^R C}_{\Sigma^{n/2} = \pm, \Sigma^R C = \pm} \tag{2.11}$$

and for n odd we have the sectors

$$H_Q^{\tilde{Q}}, H^0_{0,\pm}, H^{\Sigma^R C}_{\Sigma^R C = \pm}. \tag{2.12}$$

The case of free boundary conditions parallels the case of periodic boundary conditions ($\tilde{Q} = 0$). We have

$$\begin{aligned} &H_Q^{\mathrm{F}} = H_{n-Q}^{\mathrm{F}}, H_{0,\pm}^{\mathrm{F}}, H_{n/2,\pm}^{\mathrm{F}} \qquad n \text{ even} \\ &H_Q^{\mathrm{F}} = H_{n-Q}^{\mathrm{F}}, H_{0,\pm}^{\mathrm{F}} \qquad n \text{ odd}. \end{aligned} \tag{2.13}$$

3. Finite-size scaling, Virasoro and U(1) Kac–Moody algebras

In this section we summarise the standard lore. We start with finite-size scaling. First we consider the case of free boundary conditions (Baake *et al* (1987a, b) and references therein). We denote by $E_\Lambda^{\mathrm{F}(\pm)}(r;N)$ the energy levels of the Hamiltonian H^{F} with N sites (Λ labels the sectors); $r=0$ denotes the lowest energy level, $r=1$ the first excited state, etc. Since the Hamiltonian H^{F} is invariant under parity, $H^{\mathrm{F}(\pm)}$ denotes the parity sectors $\pm$ of the Hamiltonian. This is a space symmetry unrelated to the internal symmetries discussed in § 2. Let $E_0^{\mathrm{F}}(N)$ be the ground-state energy for a chain of N sites. This is the lowest energy level in $H_{0,+}^{\mathrm{F}(+)}$. We consider the quantities (Cardy 1984, 1986, von Gehlen and Rittenberg 1986b)

$$\mathscr{E}_\Lambda^{(\pm)}(r)=\lim_{N\to\infty}\frac{N}{\pi}(E_\Lambda^{\mathrm{F}(\pm)}(r;N)-E_0^{\mathrm{F}}(N)). \tag{3.1}$$

It is a consequence of conformal invariance that an irreducible representation $(\Delta)_{\mathrm{V}}$ of the Virasoro algebra

$$[L_m,L_n]=(m-n)L_{m+n}+\tfrac{1}{12}c(m^3-m)\delta_{m+n,0} \tag{3.2}$$

with a highest weight Δ gives the following contribution to the spectra $\mathscr{E}_\Lambda^{(\pm)}(r)$:

$$\begin{aligned}&\mathscr{E}_\Lambda^{(+)}(r)=\Delta+2r \qquad && r=0,1,\ldots\\ &\mathscr{E}_\Lambda^{(-)}(r)=\Delta+2r+1 && r=0,1,\ldots\end{aligned} \tag{3.3a}$$

or

$$\begin{aligned}&\mathscr{E}_\Lambda^{(-)}(r)=\Delta+2r \qquad && r=0,1,\ldots\\ &\mathscr{E}_\Lambda^{(+)}(r)=\Delta+2r+1 && r=0,1,\ldots\end{aligned} \tag{3.3b}$$

with a degeneracy $d(\Delta,r)$ given by the corresponding generating function:

$$\varphi_\Delta(z)=\sum_{r=0}^{\infty}z^r d(\Delta,r). \tag{3.4}$$

Since we are considering here only the case $c=1$, we have

$$\varphi_\Delta(z)=\Pi_V(z)=\prod_{m=1}^{\infty}(1-z^m)^{-1} \tag{3.5}$$

for any Δ, except for $\Delta=\frac{1}{4}t^2$ with t an integer number. In the latter case we have (Kac 1979)

$$\varphi_{t^2/4}=(1-z^{t+1})\Pi_V(z). \tag{3.6}$$

The various values of Δ which occur in the spectra are usually denoted by x_s and are called surface critical exponents.

We now consider the case of the boundary conditions compatible with the torus. Let $E_\Lambda^B(\rho,P;N)$ be the energy levels of the Hamiltonians H_Λ^B (boundary condition B and irreducible representation Λ) with N sites. P denotes the momentum (we have translational invariance in this case) and ρ the level. Let $E_0(N)$ be the ground-state energy (it is in the $H_{0,+}^0$ sector). We consider the quantities (Cardy 1986)

$$\mathscr{E}_\Lambda^B(\rho,P)=\lim_{N\to\infty}\frac{N}{2\pi}(E_\Lambda^B(\rho,P;N)-E_0(N)). \tag{3.7}$$

It is a consequence of conformal invariance that the tensor product of two irreducible representations $((\Delta)_{\mathrm{V}},(\bar{\Delta})_{\mathrm{V}})$ of two commuting Virasoro algebras gives the following contribution to the spectra (3.7):

$$\begin{aligned}&\mathscr{E}_\Lambda^B=\Delta+r+\bar{\Delta}+\bar{r}\\ &P=\Delta+r-(\bar{\Delta}+\bar{r})\end{aligned} \tag{3.8a}$$

with a degeneracy $d(\Delta, r)\ d(\bar{\Delta}, \bar{r})$. The combinations

$$x = \Delta + \bar{\Delta} \qquad s = \Delta - \bar{\Delta} \tag{3.8b}$$

are called scaling dimension and spin.

Since, we shall see, in the scaling limit the symmetry of the problem is U(1) Kac–Moody we first describe this algebra (see also the appendix). We add to the Virasoro generators L_m, the generators $T_m (m \in \mathbb{Z})$ and complete the Virasoro algebra (3.2) for $c = 1$ with the relations (Corrigan 1986)

$$[T_m, L_n] = mT_{m+n} \qquad [T_m, T_n] = m\delta_{m+n,0}. \tag{3.9}$$

An irreducible representation of the U(1) Kac–Moody algebra with $c = 1$ is given by the highest weight Δ and charge q such that

$$T_0|\Delta, q\rangle = q|\Delta, q\rangle \qquad L_0|\Delta, q\rangle = \Delta|\Delta, q\rangle \tag{3.10}$$

where Δ and q are related by $\Delta = \frac{1}{2}q^2$. All states of an irreducible representation have the same charge q and their degeneracy is independent of q and so also of Δ. It is *always* given by $\Pi_V(z)$ of equations (3.4) and (3.5). In the following, we shall denote the irreducible representations of the U(1) Kac–Moody algebra simply by (Δ), leaving the sign of q unspecified.

Since the representations of the Virasoro subalgebra $(\Delta)_V$ have a lower degeneracy for $\Delta = \frac{1}{4}t^2$ $(t \in \mathbb{Z})$ (see equation (3.6)) the U(1) Kac–Moody representations (0), $(\frac{1}{4})$, (1), $(\frac{9}{4}), \ldots$, etc, are reducible in terms of Virasoro representations. So

$$(0) = \{0\} \oplus \{1\} \tag{3.11a}$$

where

$$\{0\} = \bigoplus_{k \geqslant 0} (4k^2)_V \qquad \{1\} = \bigoplus_{k \geqslant 0} ((2k+1)^2)_V \tag{3.11b}$$

and

$$(\tfrac{1}{4}) = \bigoplus_{k \geqslant 0} (\tfrac{1}{4}(2k+1)^2)_V. \tag{3.12}$$

We now consider the U(1)-twisted Kac–Moody algebra which will turn out to be relevant in § 5. This algebra is similar to the algebra (3.9), the only difference being that instead of taking $T_m (m \in \mathbb{Z})$ we take $T_\mu (\mu \in \mathbb{Z} + \frac{1}{2})$:

$$[T_\mu, L_n] = \mu T_{\mu+n} \qquad [T_\mu, T_\nu] = \mu\delta_{\mu+\nu,0}. \tag{3.13}$$

There is no U(1) charge in this algebra. This algebra has only one irreducible representation (see the appendix):

$$(\tfrac{1}{16})_T = \{\tfrac{1}{16}\} \oplus \{\tfrac{9}{16}\} \tag{3.14a}$$

where

$$\{\tfrac{1}{16}\} = \bigoplus_{k \in \mathbb{Z}} (\tfrac{1}{16}(8k+1)^2) \qquad \{\tfrac{9}{16}\} = \bigoplus_{k \in \mathbb{Z}} (\tfrac{1}{16}(8k+3)^2). \tag{3.14b}$$

In the sums which appear in (3.14*b*) each of the representations (Δ) has the standard degeneracy $\Pi_V(z)$ given by (3.4) and (3.5).

We have now finished the necessary mathematical introduction and turn next to the physical problem.

4. Operator content of the n-state models with free boundary conditions

Before starting to consider the $n \geqslant 5$ situation, we first remind the reader about the $n = 4$ Ashkin–Teller model since the physics in the two cases is very different. The Ashkin–Teller quantum chain is defined by the Hamiltonian

$$H = \frac{1-4h}{4\sqrt{\lambda}\, h \sin(\pi/4h)} \sum_{j=1}^{N} [\sigma_j + \varepsilon\sigma_j^2 + \sigma_j^3 + \lambda(\Gamma_j\Gamma_{j+1}^3 + \varepsilon\Gamma_j^2\Gamma_{j+1}^2 + \Gamma_j^3\Gamma_{j+1})] \tag{4.1}$$

where

$$h = \frac{\pi}{4\cos^{-1}(-\varepsilon)}. \tag{4.2}$$

The phase diagram of the system (Kohmoto *et al* 1981) is shown in figure 1. It consists of a fully ordered ferromagnetic region (I), a partially ordered phase (II) separated by two Ising lines from the ferromagnetic phase (I) and the paramagnetic phase (III), an antiferromagnetic phase (IV) and a critical fan region (V). This system is massless with $c = 1$ along the self-dual line $\lambda = 1$ between the Kosterlitz–Thouless point $A(\varepsilon = -1/\sqrt{2}, h = 1)$ and the four-state Potts point $B(\varepsilon = 1, h = \frac{1}{4})$. It is also massless in the critical fan $(-1 < \varepsilon \leqslant -1/\sqrt{2}, 1 \leqslant h < \infty;\ 1/\lambda_{\max}(h) \leqslant \lambda \leqslant \lambda_{\max}(h))$ again with $c = 1$.

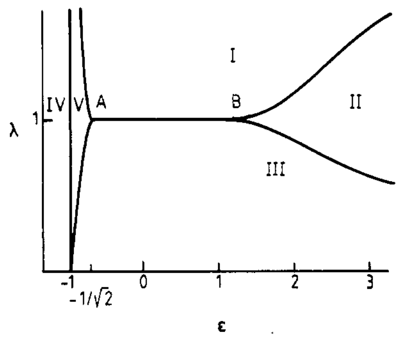

Figure 1. Phase diagram of the Ashkin–Teller quantum chain. I is the ferromagnetic region, II a partially ordered phase, III the paramagnetic phase, IV the antiferromagnetic phase and V the critical phase.

It is essential to observe that the critical exponents are dependent only on h but (inside the critical fan) not on λ. To illustrate this point, the operator content of the Ashkin–Teller model for free boundary conditions is (Rittenberg 1987, Baake *et al* 1987b, Yang 1987a)

$$\mathscr{E} = \frac{1}{4h} \bigoplus_{k\in\mathbb{Z}} (k^2) \tag{4.3}$$

(for the whole critical region with $\frac{1}{4} \leqslant h < \infty$, including the critical fan).

The situation changes completely when one considers systems with $c = 1$ and with five states or more. Here the Hamiltonian equation (1.1) has a set of couplings a_k (which correspond to ε in equation (4.1) for $n = 4$) and the critical exponents are independent of the coupling constants on the self-dual line $\lambda = 1$ but inside the critical fan they are dependent on λ. This is a reversed situation as compared to the Ashkin–Teller critical fan.

If we take a certain curve in the space of the coupling constants a_k where $c=1$ and parametrise this curve by a parameter ε, the typical situation looks as in figure 2. One has a critical fan which ends in the point A and inside the critical fan, for a given value of ε, the exponents change with λ. We will discuss in detail in the next section the variation of the critical exponents for the case of boundary conditions compatible with the torus. Here we start with the simpler case of free boundary conditions.

We begin with the observation obtained from finite-size calculations, that in the domain of the a_k where $c=1$, the lowest levels of the n-state models (we have calculated $n=5$, 6, 8 and 12) indeed turn out to be independent of the a_k and show a simple n and Q dependence. For free boundary conditions we find for the $\mathscr{E}$ of the lowest states of the various sectors at $\lambda=1$:

$$\mathscr{E}^{\mathrm{F}}_{0,-}(\text{lowest state}) = 1 \tag{4.4a}$$

$$\mathscr{E}^{\mathrm{F}}_{Q}(\text{lowest state}) = Q^2/n \qquad Q=1,\ldots,[\tfrac{1}{2}(n-1)] \tag{4.4b}$$

$$\mathscr{E}^{\mathrm{F}}_{n/2,+}(\text{lowest state}) = \mathscr{E}^{\mathrm{F}}_{n/2,-}(\text{lowest state}) = \tfrac{1}{4}n \qquad \text{for } n \text{ even.} \tag{4.4c}$$

Now this is precisely the set of lowest levels of the Ashkin–Teller model for free BC if for the coupling constant h we use the values

$$h=\tfrac{1}{4}n \tag{4.5}$$

(which in the Ashkin–Teller phase diagram are values within the critical fan region). However, the sectors in equations (4.4) appear reshuffled with respect to the sectors of the Ashkin–Teller model.

We now conjecture that the operator content summed over all sectors of the n-state models ($n \geqslant 5$) for free BC and $\lambda=1$ in the parameter domain corresponding to $c=1$

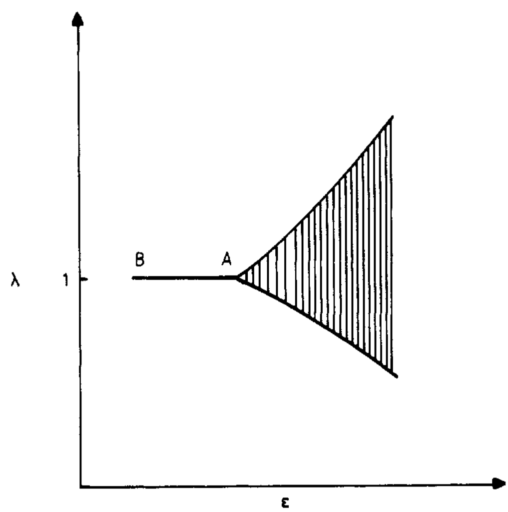

Figure 2. Part of a phase diagram of the n-state model ($n \geqslant 5$). In the space of coupling constants where $c=1$, one takes a curve parametrised by ε. In the $\varepsilon-\lambda$ plane, the system is massless with $c=1$ in the shaded area. This area has an endpoint in A. Outside the shaded area, for instance on the segment AB on the self-dual line, the central charge might be different.

is the same as that of the Ashkin–Teller model, equation (4.3), if we substitute equation (4.5):

for n even
$$\mathscr{E}^{\mathrm{F}}_{0,+} \oplus \mathscr{E}^{\mathrm{F}}_{0,-} \oplus \mathscr{E}^{\mathrm{F}}_{n/2,+} \oplus \mathscr{E}^{\mathrm{F}}_{n/2,-} \oplus \bigoplus_{\substack{Q=1\\ Q\neq n/2}}^{n-1} \mathscr{E}^{\mathrm{F}}_{Q} = \bigoplus_{k\in\mathbb{Z}} (k^2/n) \tag{4.6a}$$

and for n odd
$$\mathscr{E}^{\mathrm{F}}_{0,+} \oplus \mathscr{E}^{\mathrm{F}}_{0,-} \oplus \bigoplus_{Q=1}^{n-1} \mathscr{E}^{\mathrm{F}}_{Q} = \bigoplus_{k\in\mathbb{Z}} (k^2/n). \tag{4.6b}$$

We still have to give the distribution of the various U(1) Kac–Moody representations appearing on the right-hand side of equation (4.6) into the different sectors of our model.

For $k=0$ the right-hand side of equation (4.6) contains a series of n-independent Virasoro representations (see equation (3.11)). Inspection of the numerical results in tables 3–5 shows that n-independent levels appear only in $\mathscr{E}^{\mathrm{F}}_{0,+}$ and $\mathscr{E}^{\mathrm{F}}_{0,-}$, with the correct multiplicities for $\{0\}$ in $\mathscr{E}^{\mathrm{F}}_{0,+}$ and for $\{1\}$ in $\mathscr{E}^{\mathrm{F}}_{0,-}$.

Table 2. Normalisation factor ξ for the n-state model and different values of the coupling constants.

n	ξ	a_2	n	ξ	a_2	a_3
5	1.677 (5)	0	6	1.474 (2)	0	0
	1.466 (5)	$-\frac{1}{11}$		1.643 (3)	0.058	0.05
	1.203 (6)	$-\frac{1}{5}$		1.306 (2)	−0.058	−0.05
	0.8575 (2)	$-\frac{2}{3}$		0.249 (1)	−0.9	1
8	1.21 (1)	$a_2=\ldots$ $=a_4=0$	12	0.880 87 (3)	$a_2=\ldots=a_6=0$	

Table 3. Surface critical exponents $x_s[d]$ with a degeneracy d computed from the model for the five-state model (free BC) ($a_1=1$, $a_2=-\frac{1}{5}$). x_s(Exp) represent the numerical estimates.

$\mathscr{E}^{(+)}_{0,+}$		$\mathscr{E}^{(-)}_{0,+}$		$\mathscr{E}^{(+)}_{0,-}$		$\mathscr{E}^{(-)}_{0,-}$	
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)
2 [1]	2.002 (5)	3 [1]	2.99 (2)	2 [1]	2.037 (2)	1 [1]	1.041 (3)
4 [3]	3.98 (8); 4.11 (6)	5 [3]	4.9 (2); 5.2 (1)	4 [2]	4.03 (6); 4.03 (9)	3 [2]	3.04 (3); 3.06 (2)
	4.1 (1)		5.06 (2)			5 [4]	4.8 (2); 4.9 (1)
5 [1]	4.99 (3)			5 [1]	4.98 (1)		4.97 (1); 5.03 (3)
$\mathscr{E}^{(+)}_{1}$		$\mathscr{E}^{(-)}_{1}$		$\mathscr{E}^{(+)}_{2}$		$\mathscr{E}^{(-)}_{2}$	
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)
$\frac{1}{5}$ [1]	0.209 (1)	$\frac{6}{5}$ [1]	1.212 (5)	0.8 [1]	0.814 (1)	1.8 [1]	1.802 (4)
$\frac{11}{5}$ [2]	2.21 (1); 2.269 (1)	$\frac{16}{5}$ [3]	3.21 (2); 3.18 (3)	1.8 [1]	1.819 (1)	2.8 [1]	2.80 (2)
			3.26 (3)	2.8 [2]	2.80 (2); 2.82 (2)	3.8 [3]	3.79 (4); 3.90 (1)
							3.9 (1)
$\frac{16}{5}$ [1]	3.22 (2)						

Table 4. Same as table 3 for the six-state model (free BC) ($a_1=1$, $a_2=a_3=0$).

$\mathscr{E}^{(+)}_{0,+}$		$\mathscr{E}^{(-)}_{0,+}$		$\mathscr{E}^{(+)}_{0,-}$		$\mathscr{E}^{(-)}_{0,-}$	
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)
2 [1]	2.04 (1)	3 [1]	3.03 (6)	2 [1]	2.08 (5)	1 [1]	1.01 (1)
$\mathscr{E}^{(+)}_{1}$		$\mathscr{E}^{(-)}_{1}$		$\mathscr{E}^{(+)}_{2}$		$\mathscr{E}^{(-)}_{2}$	
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)
$\frac{1}{6}$ [1]	0.168 (1)	$\frac{7}{6}$ [1]	1.17 (5)	$\frac{2}{3}$ [1]	0.66 (2)	$\frac{5}{3}$ [1]	1.75 (1)
$\frac{13}{6}$ [2]	2.21 (5); 2.18 (4)	$\frac{19}{6}$ [3]	3.12 (3); 3.19 (2); 3.3 (1)	$\frac{8}{3}$ [3]	2.77 (4); 2.76 (4); 2.73 (4)	$\frac{11}{4}$ [4]	3.66 (2); 3.63 (1); 3.90 (1); 3.90 (4)
$\mathscr{E}^{(+)}_{3,+}$		$\mathscr{E}^{(-)}_{3,+}$		$\mathscr{E}^{(+)}_{3,-}$		$\mathscr{E}^{(-)}_{3,-}$	
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)
$\frac{3}{2}$ [1]	1.56 (1)	$\frac{5}{2}$ [1]	2.59 (3)	$\frac{3}{2}$ [1]	1.53 (5)	$\frac{5}{2}$ [1]	2.6 (1)

For the n-dependent terms we observe that an integer k can always be written as $k=nr+Q$ with $r\in\mathbb{Z}$ and $0\leq Q\leq n-1$. The expression $\bigoplus_{r\in\mathbb{Z}}(nr+Q)^2/n$ is symmetric with respect to $Q\leftrightarrow n-Q$ (see equation (2.12)) and has as its lowest level $x_s=Q^2/n$ as observed numerically in the sectors $\mathscr{E}^{\mathrm{F}}_{Q}$. If we put $Q=0$ in $(nr+Q)^2/n$ we obtain $x_s=nr^2$. This gives rise to an additional lowest state with $x_s=n$. In table 3 we observe for $n=5$ that indeed there is a state with $x_s=5$ in the sectors $\mathscr{E}^{\mathrm{F}(+)}_{0,+}$ and $\mathscr{E}^{\mathrm{F}(+)}_{0,-}$ which cannot come from {0} or {1}, respectively, because of its parity.

So we can now write the operator content of the various sectors.

(a) $n=2p+2$ (n even):

$$\mathscr{E}^{\mathrm{F}}_{0,+}=\{0\}^{+}\oplus\bigoplus_{r\geq1}(nr^2)^{+} \tag{4.7a}$$

$$\mathscr{E}^{\mathrm{F}}_{0,-}=\{1\}^{-}\oplus\bigoplus_{r\geq1}(nr^2)^{+} \tag{4.7b}$$

$$\mathscr{E}^{\mathrm{F}}_{Q}=\mathscr{E}^{\mathrm{F}}_{n-Q}=\bigoplus_{r\in\mathbb{Z}}\left(\frac{(nr+Q)^2}{n}\right) \tag{4.7c}$$

$$\mathscr{E}^{\mathrm{F}}_{n/2,+}=\mathscr{E}^{\mathrm{F}}_{n/2,-}=\bigoplus_{r\geq0}\left(\tfrac{1}{4}n(2r+1)^2\right). \tag{4.7d}$$

(b) $n=2p+1$ (n odd):

$$(4.7a)+(4.7b)+(4.7c)\text{ without }(4.7d). \tag{4.8}$$

In equations (4.7a) and (4.7b) we have used the definitions (3.11b). The superscripts $\pm$ in (4.7a) and (4.7b) indicate the relative space parity of the lowest levels of the two terms appearing in each sector.

For a detailed check of our conjecture, we have made an extensive numerical study of the n-state Hamiltonians (1.1) using finite-size scaling methods. For $n=5$, 6, 8 and 12 we have used chains up to $N=9$, 8, 7 and 6 sites, respectively. Applying the by now standard method (von Gehlen *et al* 1986) the normalisation factor ξ in

(1.1) is determined such that the sound velocity becomes equal to one. The values of ξ for particular choices of the coupling constants are listed in table 2. In each case we have checked from the finite-size corrections to the ground-state energy (Blöte *et al* 1986, Affleck 1986) that the central charge c of the Virasoro algebra is compatible with the value $c=1$. We then have determined the finite-size quantities $\mathscr{E}_\Lambda^{F(\pm)}$ defined by (3.1) and have compared them to our conjecture (4.7) and (4.8). Tables 3-5 illustrate these results by giving our numerically calculated $\mathscr{E}_\Lambda^{F(\pm)}$ together with the surface exponents x_s expected from (4.7) and (4.8). The general agreement also in non-trivial cases is good enough to decide that our conjecture is correct. The finite-size scaling estimates presented in the tables have been computed using the van den Broeck-Schwartz (1979) and Bulirsch-Stoer (1964) approximants. The errors given are very subjective and are obtained studying the variation of the approximants with the parameters which occur in the methods (for more details see Henkel and Schütz (1988)).

Before proceeding to boundary conditions compatible with the torus in the next section, we conclude the free boundary condition case by extending our conjecture (4.6) which was valid for the self-dual line $\lambda=1$ to the whole critical region with $c=1$. Numerical finite-size studies, the details of which will be given in a subsequent publication, show that we simply must generalise the right-hand side of equation (4.6) to

$$\mathscr{E}^{F} = \bigoplus_{k\in\mathbb{Z}} \left(\frac{k^2}{ng(\lambda)}\right) \tag{4.9}$$

if we abbreviate the sum of sectors on the left-hand side of (4.6) by $\mathscr{E}^F$. The function $g(\lambda)$ was discussed in (1.7).

Table 5. Same as table 3 for the eight-state model (free BC) ($a_1=1$, $a_2=a_3=a_4=0$).

$\mathscr{E}_{0,+}^{(+)}$		$\mathscr{E}_{0,+}^{(-)}$		$\mathscr{E}_{0,-}^{(+)}$		$\mathscr{E}_{0,-}^{(-)}$	
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)
2 [1]	2.000 (1)	3 [1]	2.91 (8)	2 [1]		1 [1]	1.035 (3)
$\mathscr{E}_{1}^{(+)}$		$\mathscr{E}_{1}^{(-)}$		$\mathscr{E}_{2}^{(+)}$		$\mathscr{E}_{2}^{(-)}$	
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)
$\frac{1}{8}$ [1] $\frac{17}{8}$ [2]	0.13 (1) 2.15 (1); 2.12 (3)	$\frac{9}{8}$ [1]	1.146 (2)	$\frac{1}{2}$ [1]	0.51 [1]	$\frac{3}{2}$ [1]	1.51 (1)
$\mathscr{E}_{3}^{(+)}$		$\mathscr{E}_{3}^{(-)}$		$\mathscr{E}_{4,+}^{(+)}$		$\mathscr{E}_{4,+}^{(-)}$	
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)
$\frac{9}{8}$ [1]	1.155 (5)	$\frac{17}{8}$ [1]	2.13 (2)	2 [1]	1.997 (6)	3 [1]	2.98 (1)
$\mathscr{E}_{4,-}^{(+)}$		$\mathscr{E}_{4,-}^{(-)}$					
$x_s[d]$	x_s(Exp)	$x_s[d]$	x_s(Exp)				
2 [1]	2.04 (1)	3 [1]	3.01 (3)				

5. Operator content of the n-state models with boundary conditions compatible with the torus

We consider the boundary conditions compatible with the torus. We start with the sectors which have $Z_2 \times Z_2$ symmetry (n even) or Z_2 symmetry (n odd) (see table 1). For these we find from our numerical studies, see table 7(c) (for $\lambda = 1$), that the lowest values of x (see equation (3.8b)) are $x = \frac{1}{8}, \frac{5}{8}, \frac{9}{8}, \ldots$, independent of n and independent of the couplings as long as $c = 1$. The observed multiplicities fit precisely with the assumption that these sectors are built from the representations $\{\frac{1}{16}\}$ and $\{\frac{9}{16}\}$ introduced in equation (3.14b). So we conjecture as follows.

n even ($R = 0, 1, \ldots, n-1$):

$$\begin{aligned} \mathscr{E}^{\Sigma^R C}_{\Sigma^{n/2}=+,\Sigma^R C=+} = \mathscr{E}^{\Sigma^R C}_{\Sigma^{n/2}=-,\Sigma^R C=+} &= (\{\tfrac{1}{16}\}, \{\tfrac{1}{16}\}) \oplus (\{\tfrac{9}{16}\}, \{\tfrac{9}{16}\}) \\ \mathscr{E}^{\Sigma^R C}_{\Sigma^{n/2}=+,\Sigma^R C=-} = \mathscr{E}^{\Sigma^R C}_{\Sigma^{n/2}=-,\Sigma^R C=-} &= (\{\tfrac{1}{16}\}, \{\tfrac{9}{16}\}) \oplus (\{\tfrac{9}{16}\}, \{\tfrac{1}{16}\}). \end{aligned} \tag{5.1}$$

n odd ($R = 0, 1, \ldots, n-1$):

$$\begin{aligned} \mathscr{E}^{\Sigma^R C}_{\Sigma^R C=+} &= 2(\{\tfrac{1}{16}\}, \{\tfrac{1}{16}\}) \oplus 2(\{\tfrac{9}{16}\}, \{\tfrac{9}{16}\}) \\ \mathscr{E}^{\Sigma^R C}_{\Sigma^R C=-} &= 2(\{\tfrac{1}{16}\}, \{\tfrac{9}{16}\}) \oplus 2(\{\tfrac{9}{16}\}, \{\tfrac{1}{16}\}). \end{aligned} \tag{5.2}$$

Table 6. The five-state model. Critical exponents $x[d]$ with a degeneracy d computed from the model and compared with the numerical estimates. The levels labelled by * are doubly degenerate for any number of sites ($a_1 = 1$, $a_2 = -\frac{1}{5}$).

Sector	P	$x[D]$			
$\mathscr{E}^0_{0,+}$	0	2 [1]	2.5 [2]	4 [2]	4.5 [2]
		1.99 (1)	2.48 (1)	3.793 (3)	4.5 (1)
			2.53 (5)	3.99 (1)	
	1	3 [1]	3.5 [1]	5 [3]	
		2.98 (5)	3.46 (1)	4.9 (2)	
			3.56 (1)		
	2	2 [1]	4 [2]		
		2.000 (2)	3.9 (1)		
			4.0 (1)		
$\mathscr{E}^0_1$	0	0.1 [1]	1.6 [1]	2.1 [1]	3.6 [3]
		0.1001 (3)	1.59 (2)	2.094 (5)	3.56 (2)
					3.57 (4)
					3.58 (1)
	1	1.1 [1]	2.6 [2]	3.1 [2]	4.6 [5]
		1.1002 (1)	2.58 (1)	3.1 (1)	4.56 (2)
			2.6 (1)	3.11 (3)	
$\mathscr{E}^1_1$	$-\frac{4}{5}$	1.2 [1]	1.7 [2]	3.2 [2]	3.7 [2]
		1.18 (1)	1.693 (1)	3.10 (1)	3.65 (4)
			1.725 (4)	3.18 (1)	3.67 (3)
	$\frac{1}{5}$	0.2 [1]	2.2 [1]	2.7 [2]	4.2 [4]
		0.2006 (1)	2.19 (2)	2.69 (1)	3.98 (1)
				2.72 (1)	4.1 (1)
$\mathscr{E}^1_2$	$-\frac{3}{5}$	1 [1]	1.5 [1]	3 [2]	3.5 [2]
		1.00 (1)	1.50 (2)	3.0 (1)	3.49 (5)
				3.0 (1)	3.5 (2)
	$\frac{2}{5}$	0.5 [1]	2 [1]	2.5 [1]	4 [4]
		0.5001 (1)	2.00 (2)	2.498 (4)	3.9 (1)
					3.9 (1)

Sector	P	$x[D]$		
$\mathscr{E}^0_{0,-}$	0	2.5 [2]	4 [2]	4.5 [2]
		2.501 (2)*	3.95 (2)*	4.47 (7)*
	1	1 [1]	3 [1]	3.5 [2]
		1.0002 (6)	2.96 (2)	3.46 (3)
				3.49 (1)
	2	2 [1]	4 [1]	
		1.98 (4)	3.9 (1)	
$\mathscr{E}^0_2$	0	0.4 [1]	0.9 [1]	2.4 [1]
		0.399 (4)	0.9003 (2)	2.40 (1)
		2.9 [1]	4.4 [4]	
		2.898 (3)	4.4 (1)	4.15 (2)
			4.5 (2)	
	1	1.4 [1]	1.9 [1]	3.4 [2]
		1.398 (3)	1.900 (1)	3.38 (2)
				3.39 (1)
		3.9 [3]		
		3.87 (5)	3.9 (1)	3.94 (1)
$\mathscr{E}^2_2$	$-\frac{6}{5}$	1.3 [2]	1.301 (3)	
	$-\frac{1}{5}$	1.8 [1]	2.3 [2]	3.8 [2]
		1.80 [1]	2.30 (2)	3.66 (5)
			2.31 (1)	3.73 (4)
	$\frac{4}{5}$	0.8 [1]	2.8 [2]	3.3 [4]
		0.8006 (1)	2.75 (6)	3.3 (1)
			2.8 (1)	3.3 (1)
				3.30 (3)
				3.320 (1)

Notice that if we combine the sectors together we obtain

$$2n((\tfrac{1}{16})_T, (\tfrac{1}{16})_T) \tag{5.3}$$

for n both even and odd (we have used here the notation (3.14a)). From further numerical studies we have observed that the sectors with $Z_2 \times Z_2$ (respectively Z_2) symmetry have an operator content independent of g and are given by the twisted U(1) Kac–Moody algebra.

We now consider the other boundary conditions. The finite-size numerical calculation (tables 6, 7(a) and 7(b); we have also checked the eight-state model) shows that for $\lambda = 1$ the lowest levels of the various sectors again have a very simple dependence on n, Q and $\tilde{Q}$:

$$\begin{aligned} &\mathscr{E}^0_{0,-}(\text{lowest state}) = 1(\text{doublet}) \\ &\mathscr{E}^{\tilde{Q}}_Q(\text{lowest state}) = (Q^2 + \tilde{Q}^2)/2n. \end{aligned} \tag{5.4}$$

In order to obtain (5.4) with a Gaussian form (1.4) of the primary fields which is symmetric with respect to $Q \leftrightarrow n - Q$, $\tilde{Q} \leftrightarrow n - \tilde{Q}$ (remember (2.9)), we introduce the

Table 7. Critical exponents for the six-state models ((a), (b) and (c) represent together all the sectors of the theory). The value indicated by 2.0^+ in (a) is exact by definition since the level was used to determine ξ ($a_1 = 1$, $a_2 = a_3 = 0$).

(a)

Sector	P	$x[D]$			
$\mathscr{E}^0_{0,+}$	0	2 [1]	3 [2]	4 [2]	5 [2]
		2.01 (1)	2.94 (2)	3.9 (3)	5.0 (1)
			3.02 (2)	3.92 (3)	5.1 (2)
	1	3 [1]	4 [2]		
		3.00 (3)	3.97 (5)		
			4.03 (5)		
	2	2 [1]	4 [2]	5 [4]	
		2.0^+	4.00 (5)	4.9 (1)	
			3.93 (7)		
$\mathscr{E}^0_1$	0	$\frac{1}{12}$ [1]	$\frac{25}{12}$ [2]	$\frac{49}{12}$ [8]	
		0.0833 (1)	2.085 (3)	4.0 (2)	4.13 (5)
			2.05 (5)	4.0 (2)	4.10 (2)
				4.1 (1)	4.11 (2)
				4.2 (2)	4.12 (2)
	1	$\frac{13}{12}$ [1]	$\frac{37}{12}$ [4]		
		1.0833 (1)	3.0 (1)	3.06 (3)	3.15 (3)
			3.14 (4)		
$\mathscr{E}^0_{3,+}$	0	0.75 [1]	2.75 [1]	4.75 [4]	
		0.746 (5)	2.75 (5)	4.4 (3)	4.76 (5)
				4.7 (1)	4.8 (1)
	1	1.75 [1]	3.75 [2]		
		1.74 (1)	3.6 (2)		
			3.76 (4)		

Sector	P	$x[D]$		
$\mathscr{E}^0_{0,-}$	0	3 [2]	4 [2]	5 [2]
		2.99 (4)*	3.95 (5)*	4.85 (5)*
	1	1 [1]	3 [1]	4 [2]
		0.999 (1)	3.00	3.95 (5)
				3.95 (7)
	2	2 [1]	4 [1]	5 [4]
		1.999 (5)	3.98 (5)	4.8 (1)
				4.8 (1)
$\mathscr{E}^0_2$	0	$\frac{1}{3}$ [1]	$\frac{4}{3}$ [1]	$\frac{7}{3}$ [1]
		0.3333 (1)	1.3333 (3)	2.33 (1)
		$\frac{10}{3}$ [1]	$\frac{13}{3}$ [4]	
		3.25 (3)	4.3 (1)	4.2 (1)
			4.3 (1)	4.2 (2)
	1	$\frac{4}{3}$ [1]	$\frac{7}{3}$ [1]	$\frac{10}{3}$ [2]
		1.333 (1)	2.33 (3)	3.3 (1)
				3.3 (1)
	2	$\frac{7}{3}$ [2]	$\frac{10}{3}$ [3]	
		2.33 (1)	3.28 (3)	3.3 (1)
		2.33 (1)	3.3 (1)	
$\mathscr{E}^0_{3,-}$	0	0.75 [1]	2.75 [1]	4.75 [4]
		0.748 (1)	2.78 (2)	4.7 (1)
				4.7 (1)
				4.76 (4)
				4.9 (1)
	1	1.75 [1]	3.75 [2]	
		1.75 (5)	3.7 (2)	
			3.9 (1)	

(*b*)

Sector	P	$x[D]$			
$\mathscr{E}_1^1$	$-\frac{5}{6}$	$\frac{7}{6}$ [1]	$\frac{13}{6}$ [2]	$\frac{19}{6}$ [2]	$\frac{25}{6}$ [2]
		1.17 (1)	2.12 (1)	3.2 (1)	3.9 (1)
			2.16 (2)	3.08 (2)	
	$\frac{1}{6}$	$\frac{1}{6}$ [1]	$\frac{13}{6}$ [1]	$\frac{19}{6}$ [2]	$\frac{25}{6}$ [4]
		0.1665 (1)	2.17 (3)	3.1 (1)	4.1 (1)
				3.04 (4)	4.1 (1)
					4.1 (2)
					4.1 (1)
$\mathscr{E}_{3,+}^3$	0	2.5 [1]	3.5 [2]	4.5 [2]	5.5 [3]
		2.5 (2)	3.48 (2)	4.5 (2)	5.2 (3)
			3.53 (3)	4.6 (1)	5.3 (3)
	1	1.5 [1]	3.5 [1]	4.5 [3]	
		1.502 (2)	3.50 (5)	4.2 (4)	
				4.3 (3)	
$\mathscr{E}_2^1$	$\frac{1}{3}$	$\frac{5}{12}$ [1]	$\frac{29}{12}$ [2]	$\frac{53}{12}$ [8]	
		0.4163	2.40 (3)	4.1 (2)	4.2 (2)
			2.50 (2)	4.1 (1)	4.5 (1)
				4.1 (3)	
	$\frac{4}{3}$	$\frac{17}{12}$ [1]	$\frac{41}{12}$ [4]		
		1.416 (1)	3.3 (1)	3.45 (5)	
			3.3 (3)	3.45 (5)	
$\mathscr{E}_3^2$	0	$\frac{25}{12}$ [2]	$\frac{49}{12}$ [8]		
		2.083 (5)	3.7 (2)	4.1 (1)	4.13 (3)
		2.09 (3)	3.9 (2)	4.08 (3)	4.10 (3)
			4.1 (1)	4.08 (3)	
	1	$\frac{13}{12}$ [1]	$\frac{37}{12}$ [4]		
		1.083 (1)	3.1 (1)	3.1 (1)	3.10 (3)
			3.09 (3)		

Sector	P	$x[D]$		
$\mathscr{E}_2^2$	$-\frac{1}{3}$	$\frac{5}{3}$ [1]	$\frac{8}{3}$ [2]	$\frac{11}{3}$ [2]
		1.67 (1)	2.56 (4)	3.60 (5)
			2.68 (4)	3.6 (1)
	$\frac{2}{3}$	$\frac{2}{3}$ [1]	$\frac{8}{3}$ [1]	$\frac{11}{3}$ [4]
		0.666 (1)	2.64 (4)	3.58 (5)
				3.6 (1)
				3.6 (1)
				3.7 (2)
$\mathscr{E}_{3,-}^3$	0	2.5 [1]	3.5 [2]	4.5 [2]
		2.53 (2)	3.47 (5)	4.4 (1)
			3.4 (1)	4.45 (5)
	1	1.5 [1]	3.5 [1]	4.5 [3]
		1.50 (1)	3.4 (1)	4.2 (2)
				4.3 (1)
				4.3 (2)
$\mathscr{E}_3^1$	$\frac{1}{2}$	$\frac{5}{6}$ [1]	$\frac{11}{6}$ [1]	$\frac{17}{6}$ [1]
		0.832 (1)	1.83 (3)	2.80 (5)
	$\frac{3}{2}$	$\frac{11}{6}$ [1]	$\frac{17}{6}$ [2]	$\frac{23}{6}$ [3]
		1.84 (1)	2.82 (2)	3.6 (1)
			2.84 (5)	3.8 (3)
				3.70 (3)

(*c*)

		$\mathscr{E}^C_{\Sigma^3=+,C=+}$	$\mathscr{E}^C_{\Sigma^3=-,C=+}$			$\mathscr{E}^C_{\Sigma^3=+,C=-}$	$\mathscr{E}^C_{\Sigma^3=-,C=-}$
P	$x[d]$	x(Exp)	x(Exp)	P	$x[d]$	x(Exp)	x(Exp)
0	$\frac{1}{8}$ [1]	0.124 88 (1)	0.1248 (1)	$\frac{1}{2}$	$\frac{5}{8}$ [1]	0.6249 (3)	0.625 02 (2)
	$\frac{9}{8}$ [1]	1.125 (1)	1.11 (1)		$\frac{13}{5}$ [1]	1.67 (2)	1.620 (3)
	$\frac{17}{8}$ [1]	2.12 (1)	2.11 (4)		$\frac{21}{8}$ [2]	2.67 (3); 2.70 (3)	2.62 (1); 2.60 (1)
	$\frac{25}{8}$ [4]	3.1 (1); 3.1 (1); 3.1 (1); 3.1 (1)	3.1 (1); 3.1 (2)		$\frac{29}{8}$ [4]	3.6 (3); 3.5 (3)	3.47 (3); 3.5 (1); 3.46 (5); 3.52 (5)
1	$\frac{9}{8}$ [1]	1.125 (2)	1.125 (1)	$\frac{3}{2}$	$\frac{13}{8}$ [2]	1.60 (1); 1.64 (1)	1.620 (5); 1.62 (1)
	$\frac{17}{8}$ [2]	2.15 (5); 2.12 (2)	2.1 (1); 2.20 (3)		$\frac{21}{8}$ [2]	2.63 (4); 2.70 (2)	2.55 (3); 2.55 (5)
	$\frac{25}{8}$ [2]	3.1 (1); 3.05 (4)	3.06 (5)		$\frac{29}{8}$ [3]	3.4 (2)	3.4 (2); 3.4 (1); 3.4 (3)

expressions

$$\mathscr{A}(g)=\bigoplus_{k\geq 1}\bigoplus_{l\geq 0}\left(\left(\frac{n(k+lg)^2}{4g},\frac{n(k-lg)^2}{4g}\right)\oplus\left(\frac{n(l-kg)^2}{4g},\frac{n(l+kg)^2}{4g}\right)\right) \tag{5.5}$$

$$\mathscr{B}(g)=\bigoplus_{l\geq 0}\bigoplus_{\beta\in\mathbb{Z}}\left(\frac{n(l+\frac{1}{2}+\beta g)^2}{4g},\frac{n(l+\frac{1}{2}-\beta g)^2}{4g}\right) \tag{5.6}$$

$$L(Q,\tilde{Q};g)=\bigoplus_{\alpha\in\mathbb{Z}}\bigoplus_{\beta\in\mathbb{Z}}\left(\frac{[Q+n\alpha+g(\tilde{Q}+n\beta)]^2}{4ng},\frac{[Q+n\alpha-g(\tilde{Q}+n\beta)]^2}{4ng}\right) \tag{5.7}$$

which contain sums over $(\Delta,\bar{\Delta})$ where Δ and $\bar{\Delta}$ are U(1) Kac–Moody representations with degeneracy given by $\Pi_V(z)$, equation (3.5). The function $g(\lambda)$ in (5.5)-(5.7) has the properties (1.7) and describes the behaviour as, in the critical fan, we move away from the self-dual line $\lambda=1$. The places where g occurs are restricted by the requirement that the spin $s=\Delta-\bar{\Delta}$ should not vary with λ. Whether g or g^{-1} occurs has been checked from numerical calculations which will be presented in another publication.

We can now give the operator content for the cyclic boundary conditions $\Sigma^{\tilde{Q}}$ (see table 1). We have

$$\begin{aligned}\mathscr{E}^0_{0,+}&=(\{0\},\{0\})\oplus(\{1\},\{1\})\oplus\mathscr{A}(g)\\ \mathscr{E}^0_{0,-}&=(\{0\},\{1\})\oplus(\{1\},\{0\})\oplus\mathscr{A}(g).\end{aligned} \tag{5.8}$$

For n even

$$\begin{aligned}\mathscr{E}^0_{n/2,+}&=\mathscr{E}^0_{n/2,-}=\mathscr{B}(g)\\ \mathscr{E}^{n/2}_{0,+}&=\mathscr{E}^{n/2}_{0,-}=\mathscr{B}(1/g)\\ \mathscr{E}^{n/2}_{n/2,+}&=\mathscr{E}^{n/2}_{n/2,-}=\tfrac{1}{2}L(\tfrac{1}{2}n,\tfrac{1}{2}n;g).\end{aligned} \tag{5.9}$$

For n even and n odd we have for the remaining sectors (see (2.10) and (2.11))

$$\mathscr{E}^{\tilde{Q}}_Q=L(Q,\tilde{Q};g). \tag{5.10}$$

In (5.8) we have used the definitions (3.11) for $\{0\}$ and $\{1\}$. Notice that the duality reflection $g\leftrightarrow 1/g$ interchanges Q and $\tilde{Q}$ (von Gehlen and Rittenberg 1985) and leaves $\mathscr{A}$ unchanged:

$$\begin{aligned}L(Q,\tilde{Q};g)&=L(\tilde{Q},Q,g^{-1})\\ \mathscr{A}(g)&=\mathscr{A}(g^{-1}).\end{aligned} \tag{5.11}$$

If we combine together the pairs of sectors in (5.8) and (5.9) we get the simple expressions

$$\begin{aligned}\mathscr{E}^0_0&=\mathscr{E}^0_{0,+}\oplus\mathscr{E}^0_{0,-}=L(0,0;g)\\ \mathscr{E}^0_{n/2}&=\mathscr{E}^0_{n/2,+}\oplus\mathscr{E}^0_{n/2,-}=L(\tfrac{1}{2}n,0;g)\\ \mathscr{E}^{n/2}_0&=\mathscr{E}^{n/2}_{0,+}\oplus\mathscr{E}^{n/2}_{0,-}=L(0,\tfrac{1}{2}n;g)\\ \mathscr{E}^{n/2}_{n/2}&=\mathscr{E}^{n/2}_{n/2,+}\oplus\mathscr{E}^{n/2}_{n/2,-}=L(\tfrac{1}{2}n,\tfrac{1}{2}n;g)\end{aligned} \tag{5.12}$$

and thus the whole operator content of the n-state models with all cyclic boundary conditions takes the simple form

$$\mathscr{E}=\sum_{Q=0}^{n-1}\sum_{\tilde{Q}=0}^{n-1}L(Q,\tilde{Q};g)=\bigoplus_{\alpha\in\mathbb{Z}}\bigoplus_{\beta\in\mathbb{Z}}\left(\frac{(\alpha+g\beta)^2}{4ng},\frac{(\alpha-g\beta)^2}{4ng}\right). \tag{5.13}$$

Let us discuss a few physical properties of the model. Consider first the thermal sector $\mathscr{E}^0_{0,+}$. The first excited state is always the marginal operator $(\Delta,\bar{\Delta})=(1,1)$ and one has always only one marginal except at $\lambda_{\max}(g=\frac{1}{4}n)$ or at $\lambda_{\min}=1/\lambda_{\max}(g=4/n)$ where one has two marginal operators. Actually in numerical studies when it is difficult to establish the border of the massless region (see figure 2), especially in the vicinity of the endpoint (A in figure 2), the most precise way to determine this border is just to look for the value of λ when one observes the second marginal operator. At this point we can compare the operator content conjectured by us and those exponents which were previously known. First using (3.8*b*), (5.7), (5.10) and (5.12), we derive the scaling dimensions of the primary operators in the sector $\mathscr{E}^{\tilde{Q}}_Q$, obtaining

$$x_{Q,\tilde{Q};\alpha,\beta}=\frac{(Q+n\alpha)^2}{2ng}+\frac{(\tilde{Q}+n\beta)^2g}{2n}\qquad \alpha,\beta\in\mathbb{Z}. \tag{5.14}$$

If we consider only the leading ones ($\alpha=\beta=0$) we get back the known result of Fradkin and Kadanoff (1980). The sectors described by (5.1) and (5.2) have not been considered previously since they do not correspond to the Coulomb gas.

Our numerical results which verify our conjecture on the operator content in a non-trivial way not only for the lowest levels (5.4), but which test also higher levels, are given in tables 6 and 7 for the case $\lambda=g=1$. For the five-state model we have calculated chains up to nine sites and for the six-state model up to eight sites. The tables are given for only one set of values of the coupling constants (see table 2) but the work was repeated for other values of the a_k, and it was found that the operator content is unchanged as long as we stay in the $c=1$ region. The agreement between the theoretical conjecture and the numerical estimates confirms our conjecture. Actually the check can be done in a more precise way. For $n=6$ it turns out that the system has $N=2$ superconformal invariance (Di Vecchia *et al* 1985, Waterson 1986, Boucher *et al* 1986) and thus the operator content can be compared with the known character expressions once the irreducible representations are determined. We will return to the problem of higher symmetries (beyond the U(1) Kac–Moody algebra) in these models in another publication (Baake *et al* 1987a).

6. Conclusions

The main results of this paper are given in (4.7)–(4.9) for the operator content of the n-state model with free boundary conditions and in (5.1), (5.2), (5.8)–(5.10) for the other boundary conditions. In a separate publication Suranyi (1988) shows that our results are compatible with extended modular invariance. As discussed in § 5, the lower excitations of our spectra give the previously known critical exponents for cyclical boundary conditions (Fradkin and Kadanoff (1980) and references therein). Before concluding let us present the large-n limit of the model. This corresponds to the O(2) symmetric model. Using (5.10) and (5.12) we have for finite n

$$\mathscr{E}^{\tilde{Q}}_Q=L(Q,\tilde{Q};g) \tag{6.1}$$

where $L(Q, \tilde{Q}; g)$ is given by (5.7). In order to get the large-n limit, we take

$$\tilde{Q}/n = s = \text{fixed} \tag{6.2}$$

which corresponds to the boundary condition

$$\Gamma_{N+1} = \exp(2\pi \mathrm{i} s)\Gamma_1 \qquad 0 \leqslant s < 1. \tag{6.3}$$

We will also put

$$\tilde{g} = gn \qquad 4 \leqslant \tilde{g} < \infty. \tag{6.4}$$

Taking now the limit we have

$$\mathscr{E}^s_Q = \bigoplus_{\beta \in \mathbb{Z}} \left(\frac{[Q+\tilde{g}(s+\beta)]^2}{4\tilde{g}}, \frac{[Q-\tilde{g}(s+\beta)]^2}{4\tilde{g}} \right). \tag{6.5}$$

This is the operator content of the O(2) theory in the charge sector Q with boundary condition s. From equation (6.4) we notice that, since the temperature $T \sim 1/\tilde{g}$, the temperature range of the massless phase spreads between $0 \leqslant T \leqslant T_c$. If one takes the lowest excitations ($\beta = 0$) and periodic boundary conditions ($s = 0$) in equation (6.5) one gets the scaling dimensions

$$x_Q = Q^2/2\tilde{g} \tag{6.6}$$

and we recover the known result of José *et al* (1977) (the connection between their coupling constant K_{eff} and our $\tilde{g}$ is $\tilde{g} = 2\pi K_{\mathrm{eff}}$).

Acknowledgments

One of us (VR) would like to thank USP Sao Carlos, IFT-UNESP Sao Paulo and CBPF Rio de Janeiro where part of this work was done for their hospitality and the CNPq for financial support. We would also like to thank F C Alcaraz for useful discussions.

Appendix. Irreducible representations of the U(1) Kac–Moody algebra ($c = 1$)

The untwisted U(1) Kac–Moody algebra is defined by

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}c(m^2 - m)\delta_{m+n,0} \\ &[T_m, L_n] = mT_{m+n} \qquad [T_m, T_n] = m\delta_{m+n,0} \end{aligned} \tag{A1}$$

($m, n \in \mathbb{Z}$). Its irreducible representations can be obtained using the Sugawara construction (Goddard and Olive (1986) and references therein). We take

$$L_m = \tfrac{1}{2} \sum_{r \in \mathbb{Z}} :T_{m-r}T_r: \tag{A2}$$

where:

$$:T_rT_s: = \theta(s-r)T_rT_s + \theta(r-s)T_sT_r \tag{A3}$$

and

$$\theta(x)=\begin{cases}0 & x<0\\ \frac{1}{2} & x=0\\ 1 & x>0.\end{cases} \tag{A4}$$

We get unitary representations taking the involution

$$L_m^+=L_{-m} \qquad T_m^+=T_{-m}. \tag{A5}$$

Making the change of notations

$$\begin{gathered}T_r=\sqrt{r}a_r\\ [a_r, a_s^+]=\delta_{r,s} \qquad [a_r, a_s]=0 \qquad [T_0, a_r]=0\end{gathered} \tag{A6}$$

we find

$$L_0=\tfrac{1}{2}T_0^2+\sum_{r=1}^{\infty} r a_r^+ a_r. \tag{A7}$$

The highest weight states $|\Delta, q\rangle$ (see (3.10)) correspond to the bosonic vacuum and the character of the corresponding irreducible representations is

$$\chi_{\Delta,q}(z)=\mathrm{Tr}(z^{L_0})=z^{\Delta}\Pi_V(z) \tag{A8}$$

where $\Delta=\frac{1}{2}q^2$ and $\Pi_V(z)$ is given by equation (3.5).

The twisted U(1) Kac–Moody algebra is obtained from the untwisted one (A1) taking T_μ $(\mu\in\mathbb{Z}+\frac{1}{2})$ instead of T_m $(m\in\mathbb{Z})$. We repeat the Sugawara construction and have

$$L_m=\tfrac{1}{2}\sum_{\mu\in\mathbb{Z}+1/2} :T_{m-\mu}T_\mu: +\tfrac{1}{16}\delta_{m,0} \tag{A9}$$

and

$$L_0=\sum_{\mu>0}\mu a_\mu^+ a_\mu+\tfrac{1}{16}. \tag{A10}$$

The character function corresponding to the vacuum representation is

$$\begin{gathered}\chi_{1/16}(z)=\mathrm{Tr}(z^{L_0})=z^{1/16}\varphi_{1/16}(z)\\ \varphi_{1/16}(z)=\Pi_V(z)\prod_{m=0}^{\infty}(1+z^{m+1/2})\prod_{l=0}^{\infty}(1-z^{2l}).\end{gathered} \tag{A11}$$

This gives the decomposition (3.14b) in terms of Virasoro representations. Taking the subspace of an even (odd) number of bosons one obtains $\{\frac{1}{16}\}$ ($\{\frac{9}{16}\}$) of equation (3.14b).

References

Affleck I 1986 *Phys. Rev. Lett.* **56** 746
Alcaraz F C 1986 *J. Phys. A: Math. Gen.* **19** L1085
Baake M, Christe Ph and Rittenberg V 1987a *Preprint Nordita 87/67P*
Baake M, von Gehlen G and Rittenberg V 1987b *J. Phys. A: Math. Gen.* **20** L479
—— 1987c *J. Phys. A: Math. Gen.* **20** L487
Blöte H W, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742
Boucher W, Friedan D and Kent A 1986 *Phys. Lett.* **172B** 316
Bulirsch R and Stoer J 1964 *Num. Math.* **6** 413

Cardy J L 1978 *Preprint UCSB TH-30*
—— 1984 *Nucl. Phys.* B **240** 514
—— 1986 *Nucl. Phys.* B **275** 200
Corrigan E 1986 *Phys. Lett.* **169B** 259
Di Francesco P, Saleur B and Zuber J B 1987 *Nucl. Phys.* B **285** [FS19] 454
Di Vecchia P, Peterson J L and Zheng H P 1985 *Phys. Lett.* **162B** 327
Elizur S, Pearson R B and Shigemitsu J 1979 *Phys. Rev.* B **24** 5229
Fradkin E and Kadanoff L P 1980 *Nucl. Phys.* B **170** 1
Goddard P and Olive D 1986 *Int. J. Mod. Phys.* A **1** 303
Henkel M and Schütz G 1988 *J. Phys. A: Math. Gen.* **21** 2617
José J V, Kadanoff L P, Kirkpatrick S and Nelson D R 1977 *Phys. Rev.* B **16** 1217
Kac V 1979 *Group Theoretical Methods in Physics* (*Lecture Notes in Physics* **94**) ed W Beiglbock and A Bohm (Berlin: Springer) p 444
Kadanoff L P 1979 *Ann. Phys., NY* **120** 39
Kohmoto M, den Nijs M and Kadanoff L P 1981 *Phys. Rev.* B **24** 5229
Marcu M, Regev A and Rittenberg V 1981 *J. Math. Phys.* **22** 2740
Nienhuis B 1984 *J. Stat. Mech.* **34** 731
Rittenberg V 1987 *Proc. Paris-Meudon Colloq.* ed H J de Vega and N Sánchez (Singapore: World Scientific) p 214
Saleur H 1987 *J. Phys. A: Math. Gen.* **20** L1127
Schütz G 1987 *J. Phys. A: Math. Gen.* **20** L905
Suranyi P 1988 *J. Phys. A: Math. Gen.* **21** L401
van den Broeck J M and Schwartz L N 1979 *SIAM J. Math. Anal.* **10** 658
von Gehlen G and Rittenberg V 1985 *Nucl. Phys.* B **257** [FS14] 351
—— 1986a *J. Phys. A: Math. Gen.* **19** 2439
—— 1986b *J. Phys. A: Math. Gen.* **19** L631
—— 1987 *J. Phys. A: Math. Gen.* **20** 1309
von Gehlen G, Rittenberg V and Ruegg H 1986 *J. Phys. A: Math. Gen.* **19** 107
Waterson G 1986 *Phys. Lett.* **171B** 77
Yang S-K 1987a *Nucl. Phys.* B **285** [FS19] 183
—— 1987b *Nucl. Phys.* B **285** [FS19] 639
Yang S-K and Zheng H B 1987 *Nucl. Phys.* B **285** [FS19] 410
Zamolodchikov A B and Fateev V A 1985 *Zh. Eksp. Teor. Fiz.* **89** 380 (*Sov. Phys.-JETP* **62** 215)